ME 261: Numerical Analysis

Lecture-12: Numerical Interpolation

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Inverse Interpolation

Problem : Given a table of values

Find x such that : $f(x) = y_k$, where y_k is given

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

<u>One approach:</u>

Use polynomial interpolation to obtain ***$f_n(x)$*** to interpolate the data then use Newton's method to find a solution to *x*

$$f_n(x) = y_k$$

Inverse Interpolation

Inverse interpolation:

1. Exchange the roles of x and y.

x_i	x_0	x_1	….	x_n
y_i	y_0	y_1	….	y_n

2. Perform polynomial Interpolation on the new table.

y_i	y_0	y_1	….	y_n
x_i	x_0	x_1	….	x_n

3. Evaluate

$$x = f_n(y_k)$$

Inverse Interpolation

Example

x	1	2	3
y	3.2	2.0	1.6

Problem :

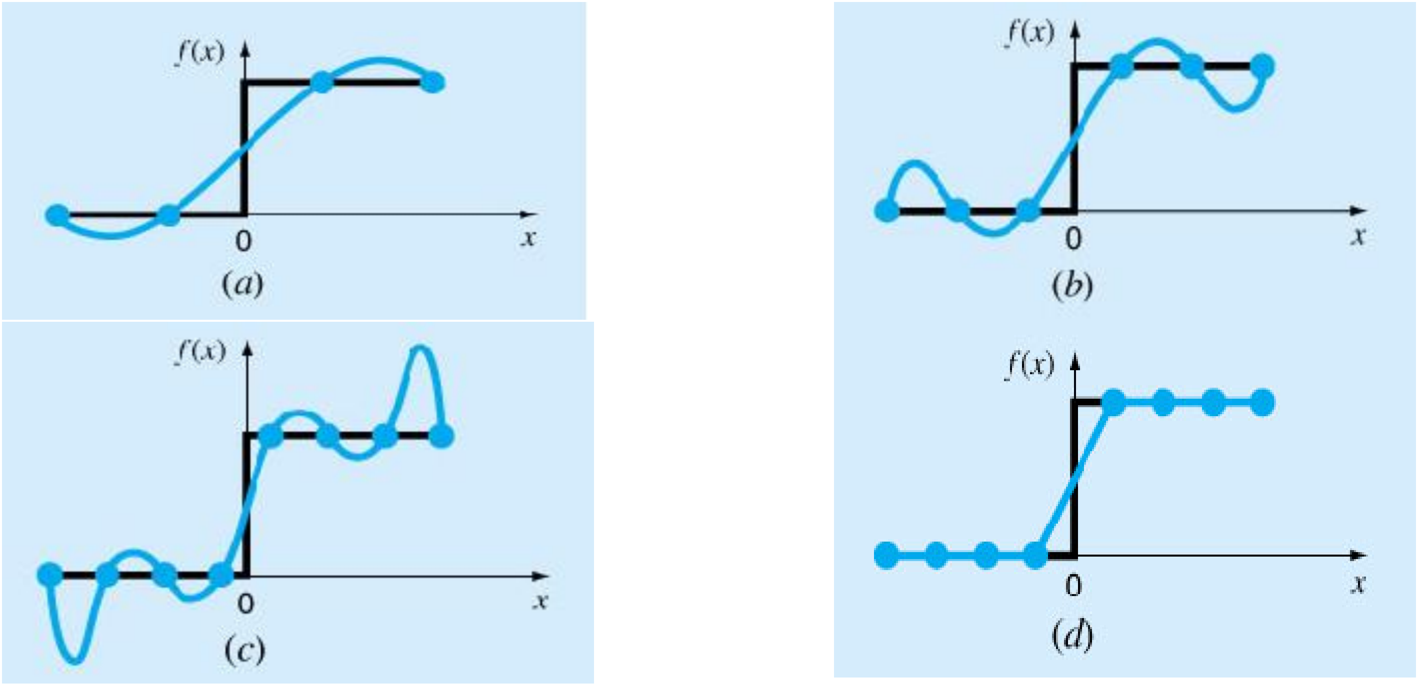
Given the table. Find x such that $f(x) = 2.5$

3.2	1	-.8333	1.0417
2.0	2	-2.5	
1.6	3		

$$x = f_2(y) = 1 - 0.8333(y - 3.2) + 1.0417(y - 3.2)(y - 2)$$

$$x = f_2(2.5) = 1 - 0.8333(-0.7) + 1.0417(-0.7)(0.5) = 1.2187$$

Why Spline Interpolation?



Apply lower-order polynomials to subsets of data points. Spline provides a superior approximation of the behavior of functions that have local, abrupt changes.

Why Splines ?

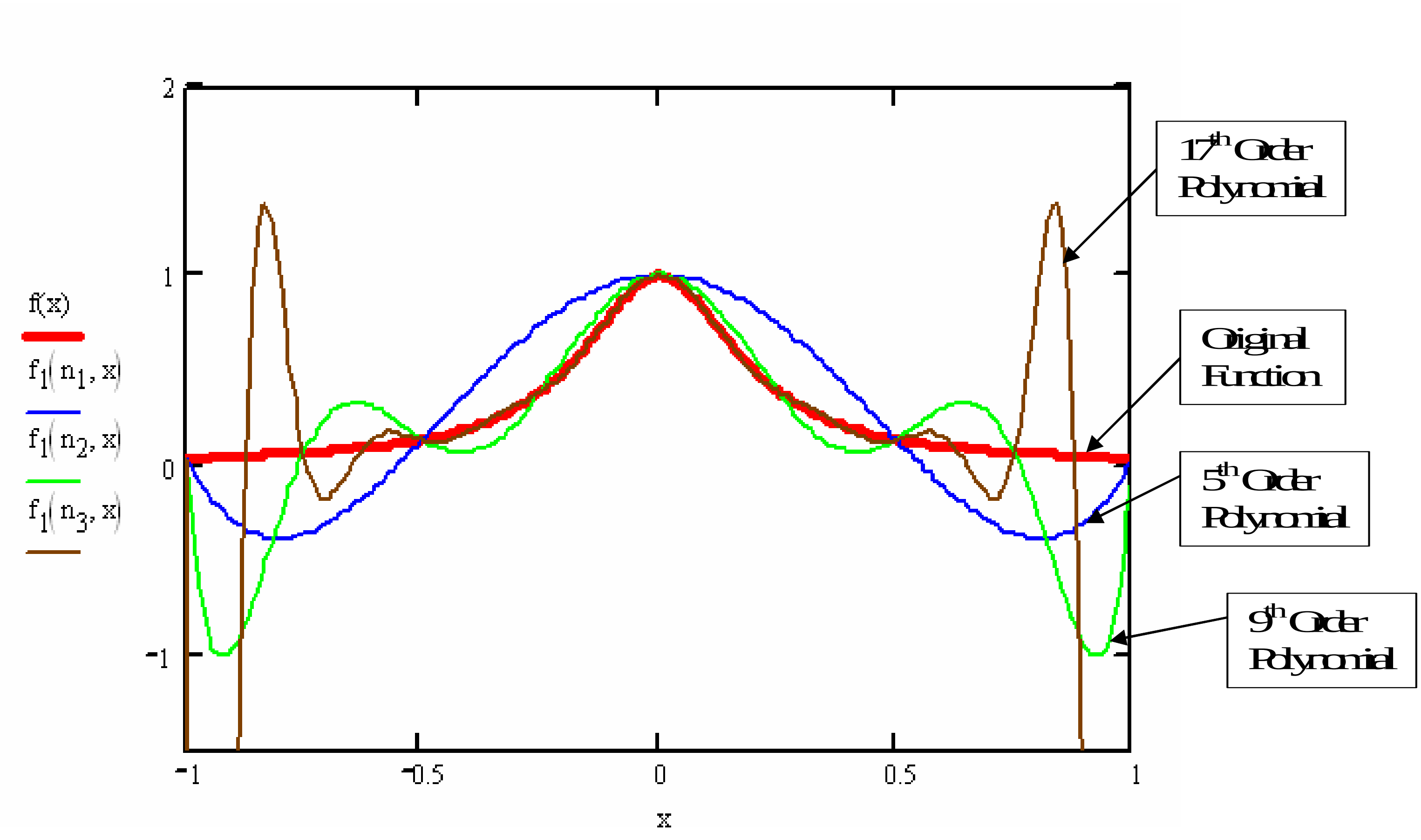
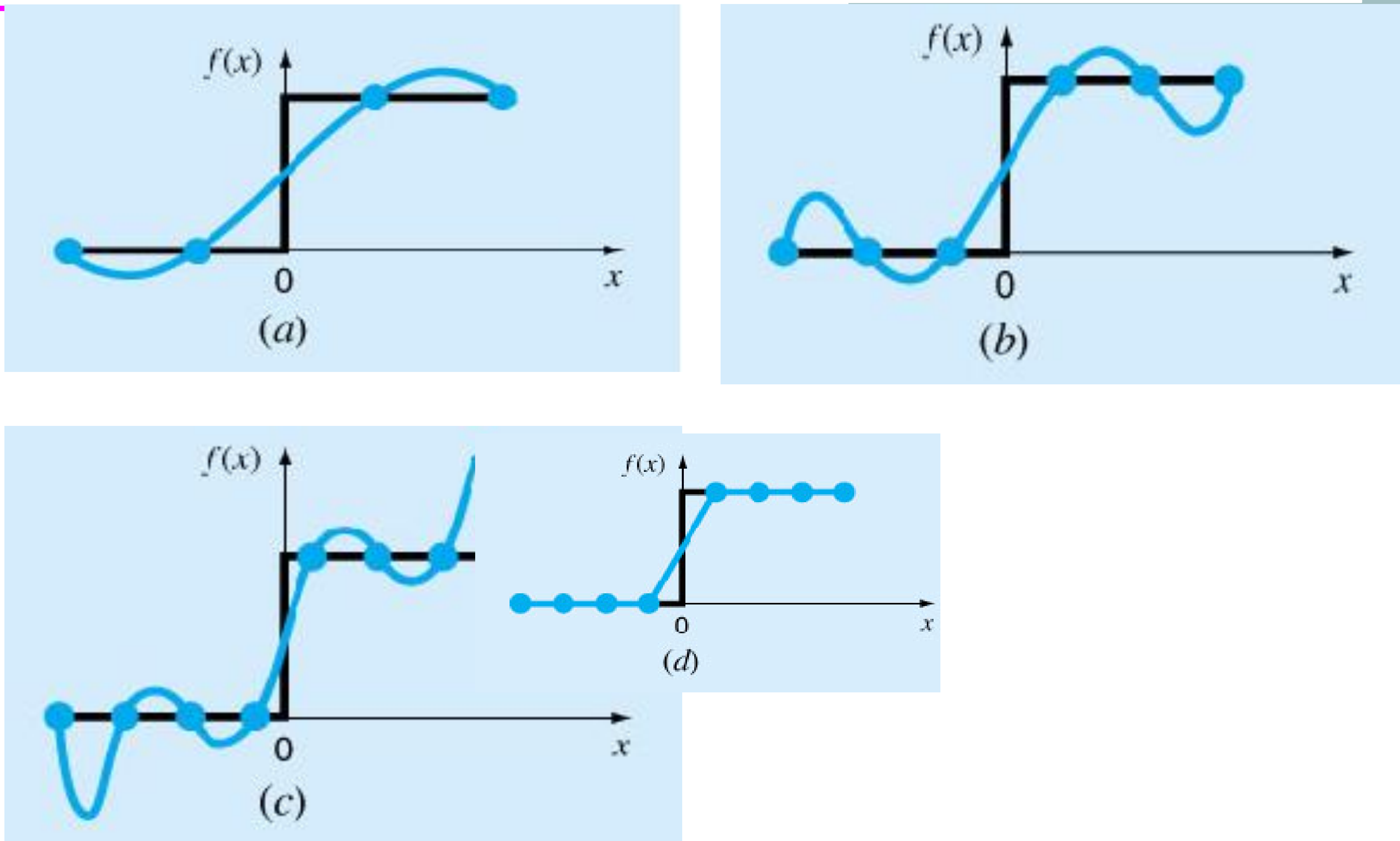


Figure : Higher order polynomial interpolation is a bad idea

Spline Interpolation

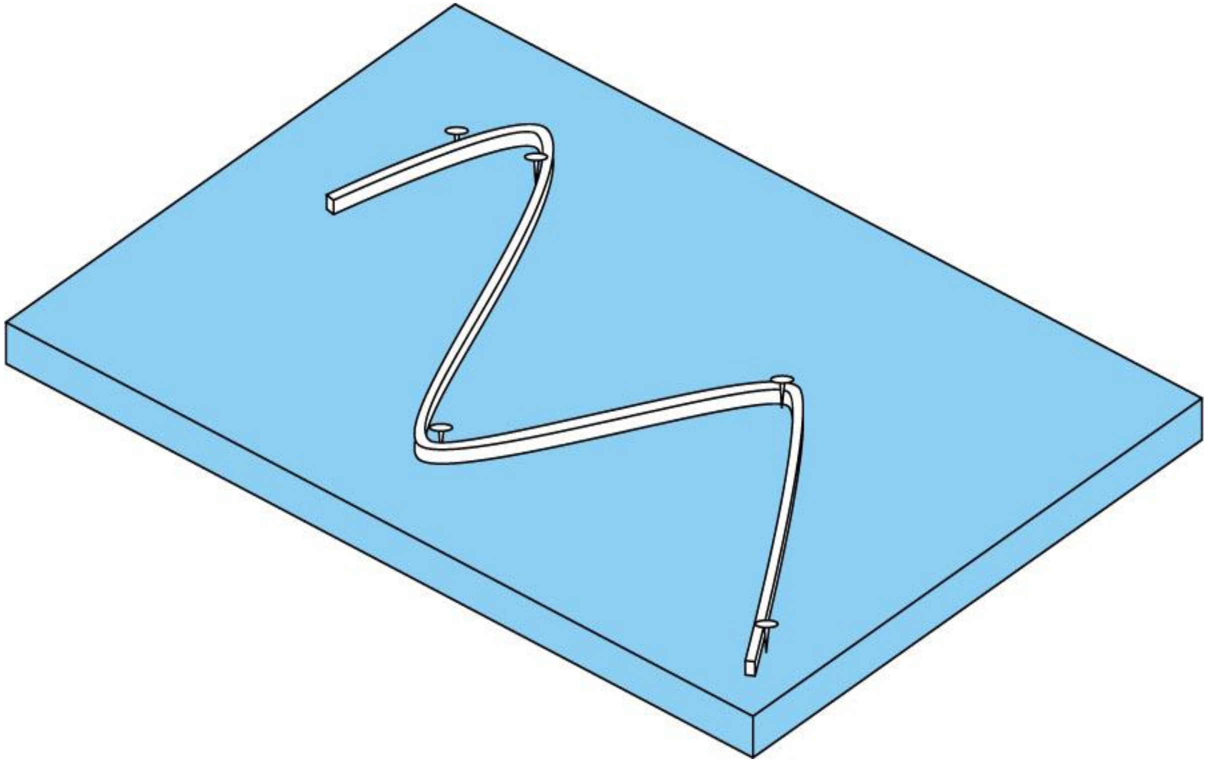
- Polynomials are the most common choice of interpolants.
- There are cases where polynomials can lead to erroneous results because of round off error and overshoot.
- Alternative approach is to apply **lower-order polynomials** to subsets of data points. Such connecting polynomials are called **spline functions.**

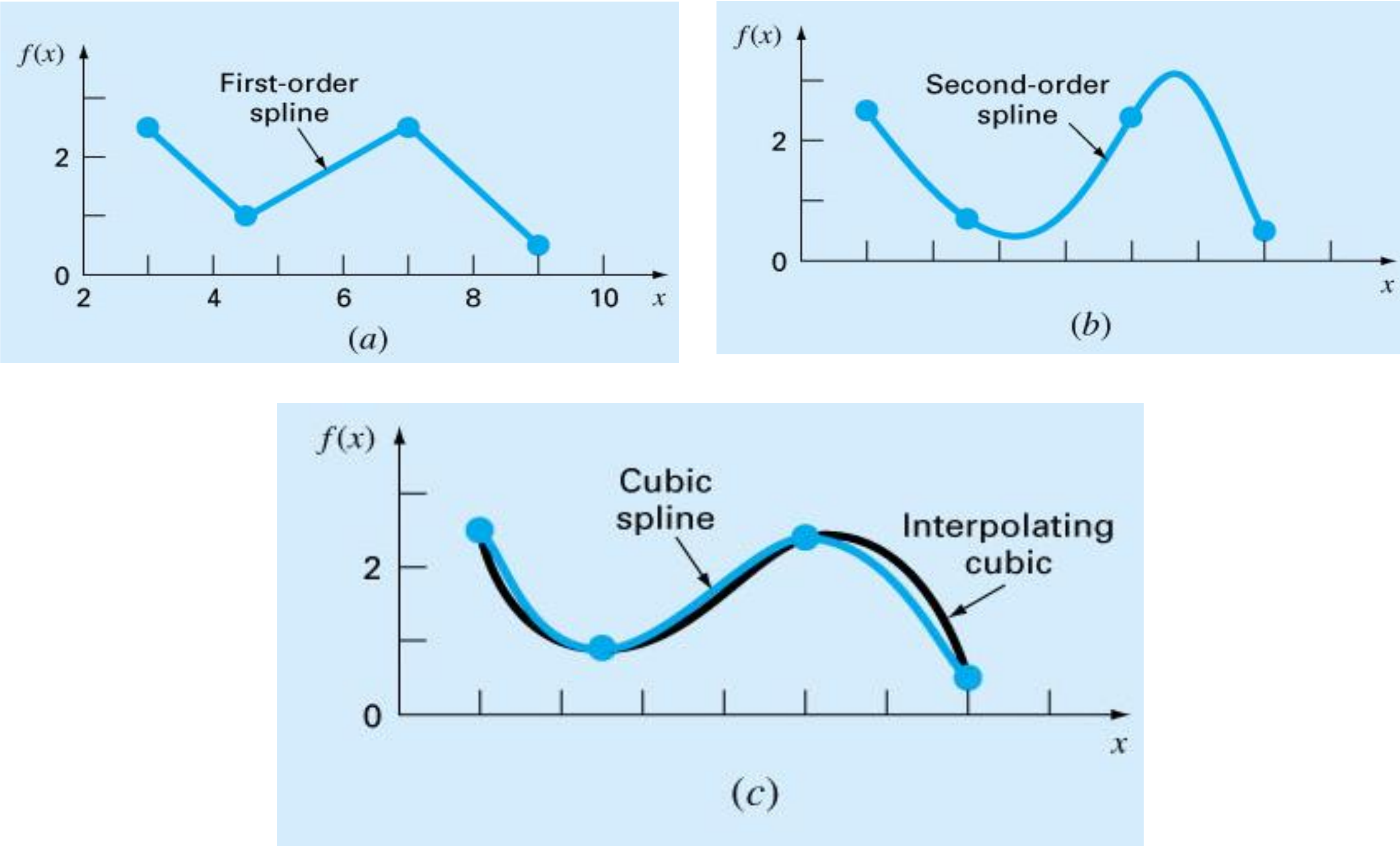


Spline provides a superior approximation of the behavior of functions that have local, abrupt changes (d).

Spline Interpolation

The concept of spline is using a thin , flexible strip (called a spline) to draw smooth curves through a set of points….**natural spline (cubic**)




f(x)
First-order
spline
2
0
2
4
6
8
10
x
(a)
f(x)
Second-order
spline
2
0
x
(b)
f(x)
Cubic
spline
Interpolating
cubic
2
0
x
(c)

Linear Spline

The first order splines for a group of ordered data points can be defined as a set of linear functions:

$$f(x) = f(x_0) + m_0(x - x_0) \qquad x_0 \le x \le x_1$$

$$f(x) = f(x_1) + m_1(x - x_1) \qquad x_1 \le x \le x_2$$

$$f(x) = f(x_{n-1}) + m_{n-1}(x - x_{n-1}) \qquad x_{n-1} \le x \le x_n$$

$$m_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Linear spline - Example

Fit the following data with **first order splines**. Evaluate the function at x = 5.

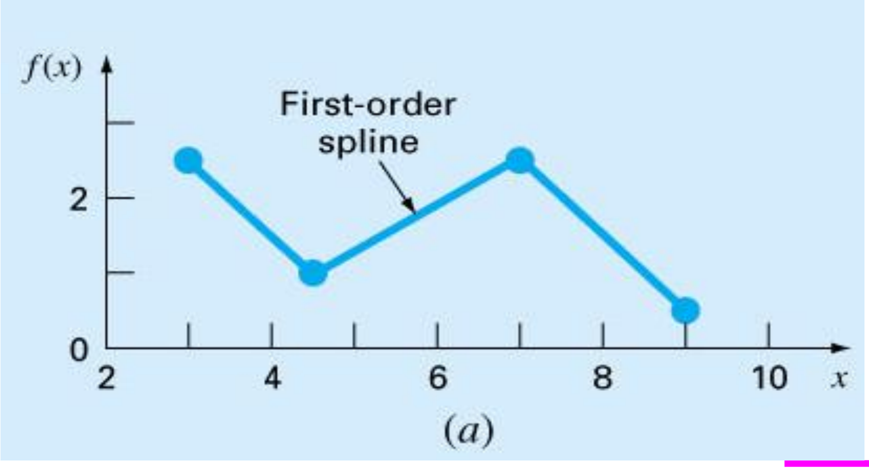
x	*f(x)*
3.0	2.5
4.5	**1.0**
7.0	**2.5**
9.0	0.5

$$m = \frac{2.5 - 1}{7 - 4.5} = 0.6$$

$$f(5) = f(4.5) + m(5 - 4.5)$$

$$= 1.0 + 0.6 \times 0.5$$

$$= 1.3$$



Linear Spline

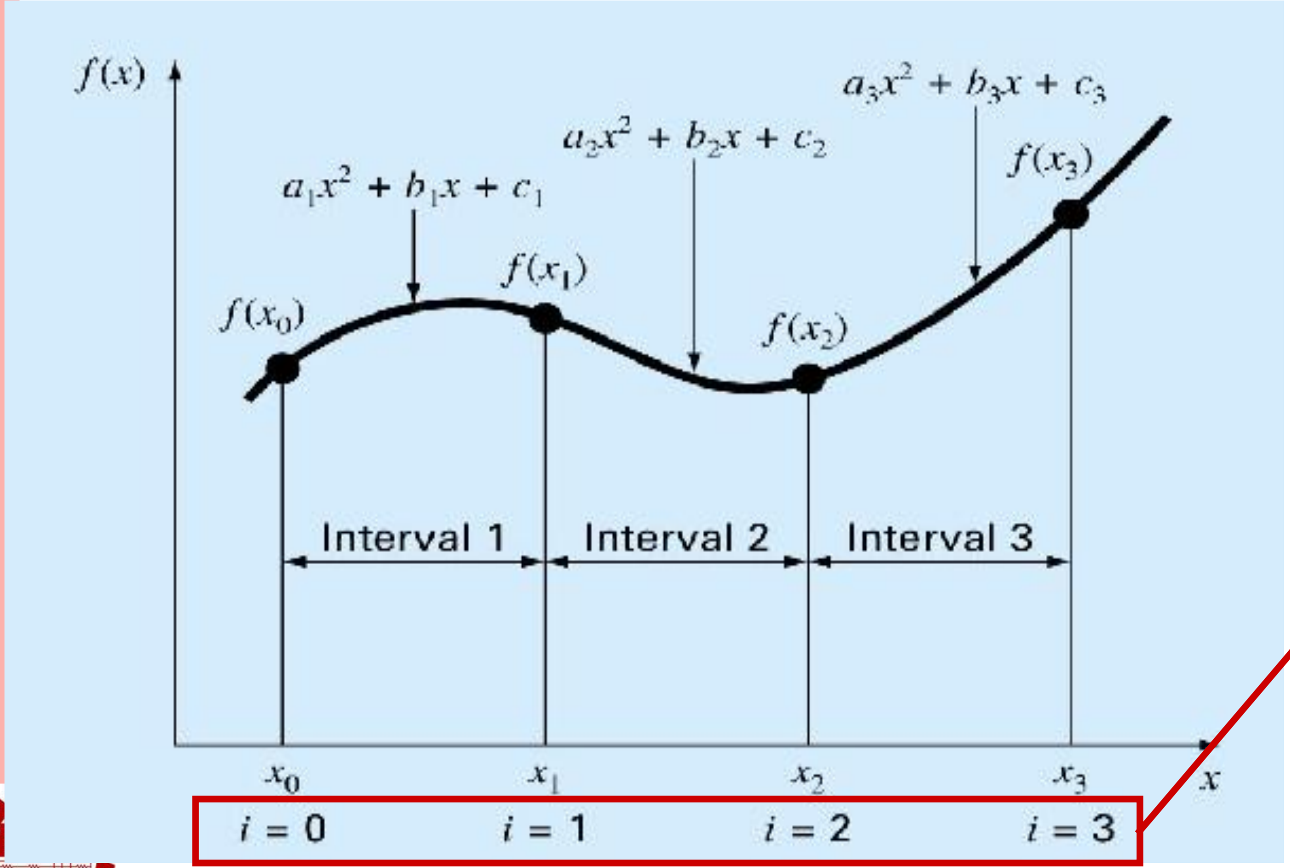
- The main **disadvantage** of **linear spline** is that they are not smooth. The data points where 2 splines meets called (a knot), the changes abruptly.

- The first derivative of the function is discontinuous at these points.

- Using **higher order polynomial splines** ensure smoothness at the knots by equating derivatives at these points.

Quadric Splines

- **Objective:** to derive a second order polynomial for each interval between data points. $f_i(x) = a_i x^2 + b_i x + c_i$
- **Terms:** Interior knots and end points



For *n*+1 data points:

- *i* = (0, 1, 2, …*n*),
- ***n*** *intervals,*
- ***3n*** *unknown constants (a's, b's and c's)*

Quadric Splines

- The function values of adjacent polynomial must be equal at the interior knots **2(n-1)**.

$$a_{i-1}x_{i-1}^2+b_{i-1}x_{i-1}+c_{i-1}=f_i(x_{i-1}) \quad i=2,3,4,\ldots,n$$

$$a_i x_{i-1}^2+b_i x_{i-1}+c_i=f_i(x_{i-1}) \quad i=2,3,4,\ldots,n$$

- The first and last functions must pass through the end points (**2**).

$$a_1 x_0^2+b_1 x_0+c_1=f(x_0)$$

$$a_n x_n^2+b_n x_n+c_n=f(x_n)$$

Quadric Splines

- The first derivatives at the interior knots must be equal (**n-1**).

$$f_i'(x) = 2a_i x + b_i$$

$$2a_{i-1} x_{i-1} + b_{i-1} = 2a_i x_{i-1} + b_i$$

- Assume that the second derivate is zero at the first point (**1**)

$$a_1 = 0$$

(The first two points will be connected by a straight line)

Quadric Splines - Example

Fit the following data with **quadratic splines.** Estimate the value at x = 5.

x	f(x)
3.0	2.5
4.5	1.0
7.0	2.5
9.0	0.5

Solutions:

There are **3** intervals (n=3), **9** unknowns.

Quadric Splines - Example

1. Equal interior points:

- **For first interior point (4.5, 1.0)**
- **For second interior point (7.0, 2.5)**

$$20.25a_1 + 4.5b_1 + c_1 = 1.0$$
$$20.25a_2 + 4.5b_2 + c_2 = 1.0$$
$$49a_2 + 7b_2 + c_2 = 2.5$$
$$49a_3 + 7b_3 + c_3 = 2.5$$

Quadric Splines - Example

- First and last functions pass the end points

For the start point **(3.0, 2.5)**

$$x_0^2 a_1 + x_0 b_1 + c_1 = f(x_0) \longrightarrow 9a_1 + 3b_1 + c_1 = 2.5$$

For the end point **(9, 0.5)**

$$x_3^2 a_1 + x_3 b_3 + c_3 = f(x_3) \longrightarrow 81 a_3 + 9b_3 + c_3 = 0.5$$

Quadric Splines - Example

- Equal derivatives at the interior knots.

For first interior point **(4.5, 1.0)**

For second interior point **(7.0, 2.5)**

$$9a_1 + b_1 = 9a_2 + b_2$$
$$14a_2 + b_2 = 14a_3 + b_3$$

$$a_1 = 0$$

$$\begin{bmatrix} 4.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20.25 & 4.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 49 & 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 49 & 7 & 1 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 81 & 9 & 1 \\ 1 & 0 & -9 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & 1 & 0 & -14 & -1 & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 2.5 \\ 2.5 \\ 2.5 \\ 0.5 \\ 0 \\ 0 \end{Bmatrix}$$

Quadric Splines - Example

Solving these 9 equations with 9 unknowns

$$a_1 = 0, \quad b_1 = -1, \quad c_1 = 5.5$$

$$a_2 = 0.64, \; b_2 = -6.76, \; c_2 = 18.46$$

$$a_3 = -1.6, \; b_3 = 24.6, \; c_3 = -91.3$$

$$f_1(x) = -x + 5.5, \qquad 3.0 \le x \le 4.5$$

$$f_2(x) = 0.46x^2 - 6.76x + 18.46, \qquad 4.5 \le x \le 7.0$$

$$f_3(x) = -1.6x^2 + 24.6x - 91.3, \qquad 7.0 \le x \le 9.0$$

Cubic Splines

Objective: to derive a third order polynomial for each interval between data points.

Terms: Interior knots and end points

$$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$$

For *n*+1 data points:

- *i* = (0, 1, 2, …*n*),
- ***n*** *intervals,*
- ***4n*** *unknown constants (a's, b's ,c's and d's)*

Cubic Splines

- The function values must be equal at the interior knots **(2n-2).**
- The first and last functions must pass through the end points **(2).**
- The first derivatives at the interior knots must be equal **(n-1).**
- The second derivatives at the interior knots must be equal **(n-1).**
- The second derivatives at the end knots are zero **(2),** (the 2nd derivative function becomes a straight line at the end points)